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Melnikov's Criterion for Nondifferentiable Weak-Noise Potentials

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The stationary probability density of Fokker–Planck models with weak noise η is asymptotically of the form $\exp[-1/\eta\varphi(q)]$. If φ is smooth, it satisfies a Hamilton–Jacobi equation at zero energy and can be interpreted as the action of an associated Hamiltonian system. Under this assumption, φ has the properties of a Liapounov function, and can be used, e.g., as a thermodynamic potential in nonequilibrium steady states. We consider systems having several attractors and show, by applying Melnikov's method to the associated Hamiltonian, that in general φ is not differentiable. A small perturbation of a model with differentiable φ leads to a nondifferentiable φ. The method is illustrated on a model used in the treatment of the unstable mode in a laser.

KEY WORDS: Weak-noise; Fokker–Planck; Hamilton–Jacobi; Melnikov function; nonequilibrium; nondifferentiable potential.

1. INTRODUCTION

The Fokker–Planck equation is a widely applicable model for the description of macroscopic fluctuating systems.[1–3] The study of the small-noise limit of these models is useful in many contexts, such as in the stability analysis of dynamical systems,[4,5] in the construction of invariant measures for strange attractors,[6] and to obtain a good approximation for weakly fluctuating systems.[7]

We are interested in the stationary probability density $P(q, \eta)$ which is a solution of the stationary Fokker–Planck equation

$$\sum_{\mu} \frac{\partial}{\partial q^\mu} \{[K^\mu(q) + \eta l^\mu(q)]\, P\} + \frac{1}{2} \sum_{\mu,\nu} \frac{\partial^2}{\partial q^\mu \partial q^\nu} [\eta Q^{\mu\nu}(q)\, P] = 0 \qquad (1.1)$$

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K^μ is the drift, ηl^μ the noise-induced drift (which is irrelevant in the weak-noise limit), and $Q^{\mu\nu}$ the diffusion tensor (symmetric, positive). The parameter η measures the strength of the noise. P can always be written as

$$P(q, \eta) = e^{-\phi(q,\eta)} \tag{1.2}$$

It has been shown[4] that under quite general conditions, for $\eta \to 0$, ϕ is asymptotically of the form

$$\phi(q, \eta) = \frac{1}{\eta}\varphi(q) + O(\eta^0) \tag{1.3}$$

The pseudopotential $\varphi(q)$ is defined variationally by minimalization of an action functional S within the set of absolutely continuous functions[4] $\tilde{q}(t)$:

$$S(\tilde{q}; q, q_0, T_1, T_2) = \int_{T_1}^{T_2} dt \sum_{\mu,\nu} Q_{\mu\nu}^{-1}[\dot{\tilde{q}}^\mu - K^\mu(\tilde{q})][\dot{\tilde{q}}^\nu - K^\nu(\tilde{q})] \tag{1.4}$$

If K^μ has a single attractor A, φ is given by

$$\varphi(q) = \min_{\substack{\tilde{q}(T_1) = q_0 \in A \\ \tilde{q}(T_2) = q}} [S(\tilde{q}(t); q, q_0)] + c \tag{1.5}$$

where q_0 lies on the attractor, T_1, T_2 are free, and c is a constant. In the case of several stationary points A_i, φ is obtained by calculating for each A_i

$$\hat{\varphi}_i(q) = \min_{\substack{\tilde{q}(T_1) = q_{0i} \in A_i \\ \tilde{q}(T_2) = q}} [S(\tilde{q}(t); q, q_{0i})] + c_i \tag{1.6}$$

and then taking the minimum over the A_i

$$\varphi(q) = \min_i \hat{\varphi}_i(q) + c' \tag{1.7}$$

The weighting constants c_i are determined by methods of Markov chains.[4] This construction shows that there is no *a priori* reason to expect $\varphi(q)$ to be differentiable. In fact we will see that in general it is not the case.[8–11] Consider the situation in which for some values of q the minimum in Eq. (1.7) lies on one of the $\hat{\varphi}_i$, while in a neighboring region it lies on another branch $\hat{\varphi}_j$. At the points where the minimum changes from one branch $\hat{\varphi}_i$ to another $\hat{\varphi}_j$ one can expect a discontinuity in the derivative of φ. However, φ cannot be arbitrarily irregular either; it can be shown[4] that φ satisfies a local Lipschitz condition which implies almost everywhere differentiability.[12,13]

$\varphi(q)$ is usually denoted as a "potential," since in the cases where it is differentiable it has the properties of a Liapounov function, and can be used as a thermodynamical potential, e.g., in nonequilibrium steady states.

If φ is differentiable (C^2), it satisfies an equation of the Hamilton–Jacobi (HJ) type at zero energy [obtained by insertion of (1.3) into (1.1) and letting $\eta \to 0$]:

$$H(q, \partial\varphi) \equiv \frac{1}{2} \sum_{\mu,\nu} Q^{\mu\nu} \frac{\partial\varphi}{\partial q^\mu} \frac{\partial\varphi}{\partial q^\nu} + \sum_\mu \frac{\partial\varphi}{\partial q^\mu} K^\mu = 0 \tag{1.8}$$

with $(\partial/\partial q)\,\varphi = 0$ on the critical points of K^μ. Under this C^2 assumption, there is a correspondence between the weak-noise asymptotics of the stochastic model and an associated Hamiltonian system: the potential φ is the action of the Hamiltonian at zero energy. This equation has been considered as the natural tool to calculate the weak-noise potential. However, when one tried to compute numerically the solution of (1.8) for nontrivial problems (e.g., with more than one attractor) one found often very pathological behaviour,[8-11] showing wild oscillations that had no reasonable physical interpretation. Graham and Tél[8,9] found an explanation by noticing that a smooth φ corresponds to smooth separatrices in the associated Hamiltonian system. This is a very unstable nongeneric property: a smooth separatrix joining two critical points of a Hamiltonian is simultaneously the unstable manifold of one point and the stable manifold of the other point. This degenerate coincidence is broken by any general small perturbation; one finds then the generic situation of homoclinic intersections of the two manifolds, and the consequent appearence of wild oscillations.[14-16] Translating back to the stochastic problem we get the following picture: Each of the $\hat{\varphi}_i$ in (1.6) is constructed piecewise from the characteristics of the HJ equation (1.8) originating at the corresponding stationary point.[4] The degenerate smooth separatrix corresponds to the situation where the different pieces $\hat{\varphi}_i$ coincide and form a single smooth function. The generic case is when the different $\hat{\varphi}_i$ intersect at nonzero angles and the separatrices have wild oscillations, which imply intersections of the characteristics. There are two kinds of nondifferentiable points: The first kind is due to the intersection of the characteristics and they are already present in the $\hat{\varphi}_i$. The second kind appears by taking the minimum in Eq. (1.7). Thus the oscillations found in the numerical solutions of the HJ equation (1.8) are an artifact. They are features of the associated Hamiltonian system, but they are not present in the weak-noise limit of the stochastic model; they are eliminated when ones takes the minimum in Eqs. (1.6), (1.7). The correspondence between the stochastic and the Hamiltonian systems is valid only when the potential φ is differentiable. Otherwise, the Hamiltonian (1.8) is only an auxiliary tool, useful for the computation of the $\hat{\varphi}_i$.[10] We remark that it is also possible to have models with a single attractor where the HJ equation does not have a unique global solution (due to intersections of the characteristics), in which case φ is also not differentiable.[4] We do not consider this case here.

V. K. Melnikov developed a method[15-17] which gives a sufficient condition for the appearence of homoclinic points and oscillating separatrices in systems with one degree of freedom subjected to a periodic perturbation. The method has been extended[18-20] to analyze general perturbations of a class of systems with n degrees of freedom, in which one degree of freedom has smooth separatrices and the other $n-1$ admit action-angle variables. In the following we adapt the method to treat the Hamiltonian (1.8) associated to the stochastic model and obtain a sufficient criterion for the nondifferentiability of the potential φ. We consider Fokker-Planck models in two dimensions with radial symmetry and show that a general perturbation that breaks this symmetry leads to a nondifferentiable potential.

2. MELNIKOV'S METHOD

We consider as unperturbed system a Fokker-Planck model with radial symmetry. In polar coordinates (r, θ) we have a θ-independent drift and diagonal diffusion tensor

$$K_0^\mu = \begin{pmatrix} K_0^1(r) \\ K_0^2(r) \end{pmatrix}; \qquad Q_0^{\mu\nu} = \begin{pmatrix} Q_0^{11}(r) & 0 \\ 0 & Q_0^{22}(r) \end{pmatrix} \tag{2.1}$$

The stationary state can be calculated explicitly[21]

$$\phi_0(r, \theta) = \phi_0(r) = \int_{r_0}^{r} dr' \left[\frac{(\partial/\partial r')\, Q_0^{11}}{Q_0^{11}} - \frac{2}{\eta} \frac{K_0^1}{Q_0^{11}} \right] \tag{2.2}$$

i.e.,

$$\varphi_0(r) = -2 \int_{r_0}^{r} dr' \, \frac{K_0^1(r')}{Q_0^{11}(r')} \tag{2.3}$$

which is independent of K_0^2 and Q_0^{22}. (K_0^2 only contributes to the drift velocity[21] $R_0^\mu = (0, K_0^2(r))$, which does not affect the stationary state.) The unperturbed potential is obviously differentiable (under standard conditions on K_0^1, Q_0^{11}). The associated Hamiltonian is

$$H_0(r, p_r, \theta, p_\theta) = F(r, p_r) + G(r, p_\theta) \tag{2.4}$$

with

$$\begin{aligned} F(r, p_r) &= \tfrac{1}{2} Q_0^{11}(r)(p_r)^2 + p_r K_0^1(r) \\ G(r, p_\theta) &= \tfrac{1}{2} Q_0^{22}(r)(p_\theta)^2 + p_\theta K_0^2(r) \end{aligned} \tag{2.5}$$

where p_r, p_θ are the canonical impulses and correspond to $\partial\varphi/\partial r$ and $\partial\varphi/\partial\theta$ in the Hamilton–Jacobi equation. We add a general perturbation εH_1

$$H^\varepsilon = H_0(r, p_r, p_\theta) + \varepsilon H_1(r, p_r, \theta, p_\theta) \tag{2.6}$$

which can represent any perturbation of K_0^μ and $Q_0^{\mu\nu}$.

We sketch now Melnikov's method adapted to (2.4)–(2.6) following essentially Ref. 18, but generalizing it to allow an r dependence in $G(r, p_\theta)$. The equations of motion for the unperturbed Hamiltonian system are

$$\dot{r} = \frac{\partial F}{\partial p_r}(r, p_r) = Q_0^{11} p_r + K_0^1 \tag{2.7a}$$

$$\dot{p}_r = -\frac{\partial F}{\partial r}(r, p_r) = -\frac{1}{2}\frac{\partial}{\partial r} Q_0^{11}(p_r)^2 - p_r \frac{\partial}{\partial r} K_0^1 \tag{2.7b}$$

$$\dot{\theta} = \frac{\partial G}{\partial p_\theta} \equiv \Omega(r, p_\theta) = Q_0^{22} p_\theta + K_0^2 \tag{2.8a}$$

$$\dot{p}_\theta = 0 \tag{2.8b}$$

The first two equations depend only on r and p_r and can be solved independently of the other two. Inserting then $r(t)$ into Eq. (2.8a), $\theta(t)$ can be obtained by integration.

We make the following assumptions on F and G:

(i) We assume that the system (2.7) contains several critical points joined by separatrices [which are smooth since (2.7) is effectively one dimensional]. In the analysis one considers separately each separatrix joining two critical points (heteroclinic orbit). One only needs to consider the system in a neighborhood of each separatrix.

(ii) We assume that in a neighborhood (in p_θ) of the considered orbits

$$\Omega(r, p_\theta) \equiv \frac{\partial}{\partial p_\theta} G(r, p_\theta) \neq 0 \tag{2.9}$$

which guarantees the (local) existence of $G^{-1}(r, \alpha)$, the inverse of G with respect to p_θ. Further, $\theta(t)$ is a monotone function of t.

One proceeds in three steps:

(1) The autonomous system (2.7), (2.8) with two degrees of freedom at the fixed energy $E = 0$ can be reduced to a nonautonomous single degree of freedom system, essentially by using energy conservation and letting θ take the role of a time.[18]

(2) The standard Melnikov method can be applied to the reduced system.

(3) The criterion is then translated back to the original variables.

2.1. Reduction

(a) Since the total energy is conserved, the equation

$$H^{\varepsilon}(r, p_r, \theta, p_\theta) = E \tag{2.10}$$

can be used to eliminate p_θ. The condition (2.9) guarantees for small ε the invertibility of (2.10) with respect to p_θ in any compact subset where $\Omega(r, p_\theta) \neq 0$:

$$p_\theta = L^{\varepsilon}(r, p_r, \theta, E) \tag{2.11}$$

L^{ε} can be developed in powers of ε:

$$L^{\varepsilon} = L_0(r, p_r, E) + \varepsilon L_1(r, p_r, \theta, E) + O(\varepsilon^2) \tag{2.12}$$

with

$$L_0 = G^{-1}(r, E - F(r, p_r)) \tag{2.13}$$

$$L_1 = \frac{-H_1(r, p_r, \theta, L^0(r, p_r, E))}{\Omega(r, L^0(r, p_r, E))} \tag{2.14}$$

(b) For small ε, $\theta(t)$ is monotone [from condition (2.9)] and can be inverted $t = t(\theta)$. The equations for r and p_r become

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{dr}{dt}\Big/\frac{d\theta}{dt} = \frac{\partial H^{\varepsilon}}{\partial p_r}\Big/\frac{\partial H^{\varepsilon}}{\partial p_\theta} \\ \frac{dp_r}{d\theta} &= \frac{dp_r}{dt}\Big/\frac{d\theta}{dt} = -\frac{\partial H^{\varepsilon}}{\partial r}\Big/\frac{\partial H^{\varepsilon}}{\partial p_\theta} \end{aligned} \tag{2.15}$$

From implicit differentiation of (2.10) we get the identities

$$\begin{aligned} \frac{\partial H^{\varepsilon}}{\partial r} + \frac{\partial H^{\varepsilon}}{\partial p_\theta}\frac{\partial L^{\varepsilon}}{\partial r} &= 0 \\ \frac{\partial H^{\varepsilon}}{\partial p_r} + \frac{\partial H^{\varepsilon}}{\partial p_\theta}\frac{\partial L^{\varepsilon}}{\partial p_r} &= 0 \end{aligned} \tag{2.16}$$

which substituted into (2.15) implies

$$\begin{aligned}\frac{dr}{d\theta} &= -\frac{\partial L^{\varepsilon}}{\partial p_r}\\ \frac{dp_r}{d\theta} &= \frac{\partial L^{\varepsilon}}{\partial r}\end{aligned} \tag{2.17}$$

Thus θ takes the role of a time, and $-L^{\varepsilon}$ the role of a Hamiltonian, with an autonomous part $-L_0$ and a periodic time-dependent perturbation $-\varepsilon L_1 + O(\varepsilon^2)$. For $\varepsilon = 0$ Eqs. (2.17) reduce to the system corresponding to F, and thus (2.17) has critical points linked by an heteroclinic orbit (separatrix). Thus we can apply Melnikov's method to the system $-L^{\varepsilon}$.

2.2. Melnikov's Criterion

We have an unperturbed system with Hamiltonian $L_0(r, p_r)$ having an heteroclinic orbit

$$r = \bar{r}(\theta - \theta_0), \qquad p_r = \bar{p}_r(\theta - \theta_0) \tag{2.18}$$

at an energy l_0. θ_0 is the initial time at which the particle is at a given initial point. We perturb it by $\varepsilon L_1(r, p_r, \theta) + O(\varepsilon^2)$ which is 2π periodic in θ. We define the Melnikov function

$$M(\theta_0) = \int_{-\infty}^{\infty} d\theta \{L_0, L_1\}(\theta - \theta_0) \tag{2.19}$$

where $\{\ ,\ \}$ is a Poisson bracket evaluated on the unperturbed heteroclinic orbit (2.18). M is a measure in first order in ε of the distance between the stable and unstable manifolds.[15-17] Melnikov's theorem states that if $M(\theta_0)$ has simple zeros, then the perturbed system has transversally intersecting stable and unstable manifolds, i.e., wildly oscillating separatrices.

2.3. Translation in Terms of H_0, H_1

First we express $\{L_0, L_1\}$ in terms of F, G, H_1:

$$\{L_0, L_1\} = \frac{1}{\Omega^2}\{F, H_1\} + \frac{1}{\Omega^3} H_1 \frac{\partial F}{\partial p_r}\frac{\partial \Omega}{\partial r} - \frac{1}{\Omega}\frac{\partial H_1}{\partial p_r}\frac{\partial G^{-1}}{\partial r} \tag{2.20}$$

Remark that in the special case when G does not depend on r, this expression reduces to $1/\Omega^2\{F, H_1\}$. We have to evaluate (2.20) at the unperturbed separatrix: $r(t)$ and $p_r(t)$ are given from Eq. (2.7), p_θ is constant, and determined by the following consideration. Since the problem comes from a stochastic model, the total unperturbed energy is zero: $H_0 = F + G = 0$; and since F has the heteroclinic orbit at $F = 0$, we get the condition

$$G = \tfrac{1}{2} Q_0^{22} (p_\theta)^2 + p_\theta K_0^2 = 0 \tag{2.21}$$

We can choose one of the two roots of (2.21). We take, e.g., $p_\theta = 0$ which gives simpler formulas, since

$$\frac{\partial G^{-1}}{\partial r}(r, p_\theta = 0) = 0 \tag{2.22}$$

$$\Omega(r, p_\theta = 0) = K_0^2(r) \tag{2.23}$$

After changing variables $\theta \to t$, $d\theta = \Omega dt$ the Melnikov function becomes

$$M(t_0) = \int_{-\infty}^{\infty} dt \left[\frac{1}{K_0^2} \{F, H_1\} + \frac{1}{(K_0^2)^2} H_1 \frac{\partial F}{\partial p_r} \frac{\partial K_0^2}{\partial r} \right] \tag{2.24}$$

where the bracket is evaluated at $r = \bar{r}(t - t_0)$, $p_r = \bar{p}_r(t - t_0)$ of Eq. (2.18), and

$$p_\theta = 0, \qquad \theta(t - t_0) = \int_{t_0}^{t} d\tilde{t}'\, K_0^2(r(\tilde{t})) + \theta_0 \tag{2.25}$$

2.4. Melnikov Criterion for Nondifferentiable Potential

If the Melnikov function (2.24) has simple zeros, it implies that the potential φ of the perturbed system will be nondifferentiable.

We remark that since the Melnikov function is constructed as the distance between the stable and unstable manifolds in first order in ε,[15–17] it is easy to see that for Hamiltonian perturbations M always has zeros. The question is if the zeros are simple or degenerate, or if M is identically zero. It is clear from Eq. (2.24) that the generic case will be with simple zeros, since the condition for degeneracy is very restrictive. Assuming that a particular perturbation should satisfy it, it can be broken by the slightest modification of the perturbation. We can conclude therefore that the generic case of the considered models will have nondifferentiable weak-noise potentials.

3. EXAMPLE

As an example we treat a model of the type that was discussed in Refs. 8–10 using numerical and other approximation methods. We consider a Fokker–Planck equation with drift

$$K^{\mu} = \begin{pmatrix} r - r^3 + \varepsilon f(r) \cos\theta \\ \omega \end{pmatrix} \tag{3.1}$$

and diffusion tensor

$$Q^{\mu\nu} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{3.2}$$

$f(r)$ is some given function characterising the perturbation. The unperturbed model has a limit cycle with radial symmetry. It has been widely used, e.g., in the treatment of the unstable mode in a laser.[22,1,2] The unperturbed potential is given by

$$\varphi_0 = \frac{1}{a}\left(-r^2 + \frac{1}{2} r^4\right) \tag{3.3}$$

The associated Hamiltonian is $H = H_0 + \varepsilon H_1$ with

$$\begin{aligned} H_0 &= F(r, p_r) + G(p_\theta) = \frac{a}{2}(p_r)^2 + p_r[r - r^3] + \frac{b}{2}(p_\theta)^2 + p_\theta \omega \\ H_1 &= p_r f(r) \cos\theta \end{aligned} \tag{3.4}$$

$F(r, p_r)$ has two stationary points at zero energy ($r \geqslant 0$), $(1, 0)$ and $(0, 0)$, linked by the separatrices shown in Fig. 1. The separatrices at $p_r \equiv 0$ are

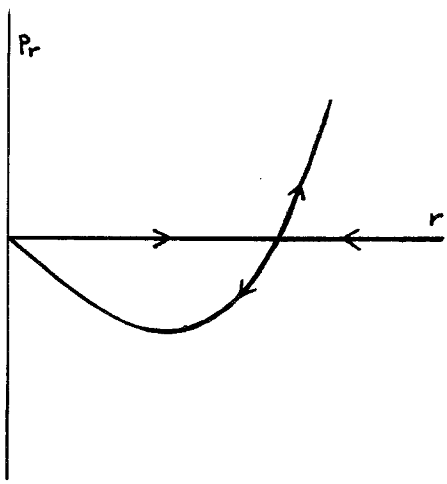

Fig. 1. Separatrices of $F = (1/2)(p_r)^2 + p_r(r - r^3)$.

always present (and stable) owing to the special form of the Hamiltonian. We will analyze the behavior of the separatrix joining the points $(1, 0)$ and $(0, 0)$ with $p_r \neq 0$.

The time evolution on the unperturbed separatrix is given by Eqs. (3.5), (3.7), (3.8) below: From the condition $F(r, p_r) = 0$, $p_r \neq 0$ we get

$$p_r = \frac{2}{a}(r^3 - r) \tag{3.5}$$

which together with the equation of motion

$$\dot{r} = \frac{\partial F}{\partial p_r} = ap_r + (r - r^3) \tag{3.6}$$

leads to

$$t - t_0 = \frac{1}{2} \ln\left(\frac{1}{r^2} - 1\right) + \gamma_0 \tag{3.7}$$

where γ_0 is a constant, independent of t_0. For θ, p_θ we set according to Eq. (2.24) and $K_0^2 = \omega$

$$p_\theta = 0, \qquad \theta = \omega t \tag{3.8}$$

The Poisson bracket

$$\begin{aligned}\{F, H_1\} &= p_r \cos\theta \left[(1 - 3r^2) f - (ap_r + r - r^3)\frac{\partial}{\partial r} f\right] \\ &= \frac{2}{a}(r^3 - r)\cos\omega t \left[(1 - 3r^2) f + (r - r^3)\frac{\partial f}{\partial r}\right]\end{aligned} \tag{3.9}$$

can be expressed using (3.7) as a function only of r. By changing variables $t \to r$ in the integral that defines the Melnikov function one gets

$$\begin{aligned}M(t_0) &= \frac{1}{\omega}\int_{-\infty}^{\infty} dt \{F, H_1\}(t - t_0) = \frac{1}{\omega}\int_1^0 dr \frac{1}{r^3 - r}\{F, H_1\}(r, t_0) \\ &= \frac{2}{a\omega}\int_1^0 dr \cos\left\{\omega\left[t_0 + \gamma_0 + \frac{1}{2}\ln\left(\frac{1}{r^2} - 1\right)\right]\right\} \\ &\quad \times \left[(1 - 3r^2) f + (r - r^3)\frac{\partial}{\partial r} f\right]\end{aligned} \tag{3.10}$$

The dependence on t_0 can be readily analyzed by writing

$$M(t_0) = A \cos \omega t_0 + B \sin \omega t_0 \tag{3.11}$$

where A, B are constants given by

$$A = \frac{2}{a\omega} \int_1^0 dr \cos\left[\omega\left(\gamma_0 + \frac{1}{2}\ln\left(\frac{1}{r^2} - 1\right)\right)\right] \times \left[(1-3r^2) f + (r - r^3)\frac{\partial f}{\partial r}\right] \tag{3.12a}$$

$$B = -\frac{2}{a\omega} \int_1^0 dr \sin\left\{\omega\left[\gamma_0 + \frac{1}{2}\ln\left(\frac{1}{r^2} - 1\right)\right]\right\} \times \left[(1-3r^2) f + (r - r^3)\frac{\partial f}{\partial r}\right] \tag{3.12b}$$

$M(t_0)$ will have zeros at the points that satisfy

$$\tan \omega t_0 = -\frac{A}{B} \tag{3.13}$$

The condition for degeneracy $(d/dt_0)\, M = 0$ gives

$$\tan \omega t_0 = \frac{B}{A} \tag{3.14}$$

(3.13) and (3.14) imply $A = 0 = B$. Thus $M(t_0)$ has an infinite number of zeros, that are simple unless the coefficients A and B vanish identically.

By making another coordinate transformation

$$t' = \gamma_0 + \frac{1}{2}\ln\left(\frac{1}{r^2} - 1\right) \tag{3.15}$$

one can express A and B as Fourier transforms of a function $g(t')$

$$A = \int_{-\infty}^{\infty} dt' \cos \omega t' g(t'), \qquad B = -\int_{-\infty}^{\infty} dt' \sin \omega t' g(t') \tag{3.16}$$

$$g(t') = (r^3 - r)\left[(1-3r^2) f + (r - r^3)\frac{\partial f}{\partial r}\right] \tag{3.17}$$

Thus A and B are zero if and only if $g(t')=0$, i.e.,

$$(1-3r^2)\,f+(r-r^3)\frac{\partial f}{\partial r}=0 \tag{3.18}$$

The only solutions of this equation for $r\in[0,1]$ are

$$f(r)=\frac{\text{const}}{|r-r^3|} \tag{3.19}$$

which have singularities at $r=0$ and $r=1$. Thus, if we consider only perturbations without singularities we can conclude that any perturbation of the considered form (3.1) will lead to a nondifferentiable potential φ. If instead of $r-r^3$ one takes a more general K_0^1, the preceding calculation can be essentially repeated, and the conclusion is also valid in this case.

4. CONCLUDING REMARKS

It is clear that the foregoing procedure can also be applied to problems with two variables having some other symmetry (i.e., not necessarily radial). The method can be easily extended to treat models with n variables, under the assumption that in the unperturbed problem one degree of freedom has heteroclinic orbits and the other $n-1$ admit action-angle variables. One can use a vectorial analog of the Melnikov function, which was introduced by Holmes and Marsden.[19] The more general case in which several of the unperturbed degrees of freedom have heteroclinic orbits would require a further generalization. There is however little doubt that the generic case always has transversely intersecting stable and unstable manifolds and thus a nondifferentiable associated potential φ.

In this context the two following questions appear to be relevant: how to determine φ explicitly, and how to use and interpret physically the nondifferentiable potentials. There are several proposals of numerical methods to calculate φ. In one approach[10] based on Eq. (1.7), one calculates the different $\hat{\varphi}_i$ as solutions of the Hamilton–Jacobi equation and one takes the minimum at each point. Another approach is based on the fact that the potential φ can be identified with the so-called "viscosity solution" of the Hamilton–Jacobi equation.[23] Finite element algorithms have been developed[24] with good convergence to the viscosity solution. The precise role of nondifferentiable potentials in nonequilibrium statistical mechanics is yet to be established. φ gives often a good approximation for the probability density. But when it is not smooth, it is not a Liapounov function and its possible use as generalized thermodynamic potential must still be investigated.

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